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CERTAIN TRANSMISSION AND REFLECTION THEOREMS

by

VIC TWERSKY

JULY 1953

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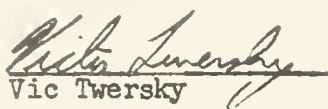
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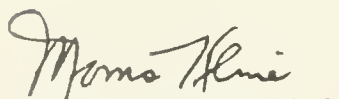
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Abstract

The well-known relation between the total energy cross-section of a scatterer and its forward scattered amplitude is extended to obtain an approximate transmission coefficient for a uniform planar distribution of parallel cylinders. An analogous reflection theorem for an arbitrary cylindrical boss on a perfectly reflecting plane is then derived; here the total cross-section of the boss is related to the scattering amplitude in the specular direction of reflection of the plane. This is extended to obtain an approximate reflection coefficient for a uniform distribution of cylindrical bosses on a plane.

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1. Introduction

In this paper we treat certain theorems which are of interest in connection with the scattering and reflection of electromagnetic and acoustic waves.

In Section 2, which serves primarily to introduce notation, we consider the scattering of a plane wave by a cylinder of arbitrary cross-section and arbitrary physical constants. We present the well-known relation¹ that the real part of the forward scattered amplitude is proportional to the total "energy cross-section" per unit length of cylinder, which is defined as the power the cylinder dissipates as heat and as scattered radiation per unit length. We then derive a corollary of this theorem and apply it to obtain an approximate transmission coefficient for a uniform planar distribution of identical cylinders.

In Section 3 we consider the analogous reflection problem for an arbitrary cylindrical protuberance on a smooth perfectly reflecting plane. We derive a theorem stating that the total cross-section is proportional to the real part of the scattering amplitude in the direction of specular reflection of the plane. As in the transmission case, we then derive a corollary and apply it to obtain an approximate reflection coefficient for a uniform distribution of identical bosses.

2. The Transmission Case

The two-dimensional problem of the scattering of a plane wave by a cylinder of arbitrary cross-section and arbitrary dielectric properties (as in Fig. 1) may be formulated as follows:

-
1. This theorem was first proved by several writers for a variety of special cases, e.g., C.H. Pappas, J. Appl. Phys. 21, 318 (1950). The most general derivation is, we believe, due to M. Lax, Phys. Rev. 18, 306 (1950).

We seek a solution of the reduced wave equation $(\nabla^2 + k^2)\Psi = 0$ for $k = 2\pi/\lambda$ which satisfies prescribed boundary conditions on the cylinder, and is of the form

$$(1) \quad \Psi = \Psi_i + \Psi_c,$$

where Ψ_i is a plane wave incident at an angle α :

$$(2) \quad \Psi_i(\theta, \alpha) = e^{ikr \cos(\theta - \alpha)},$$

and where Ψ_c , the scattered wave, is an outgoing wave as $r \rightarrow \infty$; the suppressed time factor is assumed to be $\exp(-i\omega t)$. The coordinates we use are illustrated in Fig. 1.

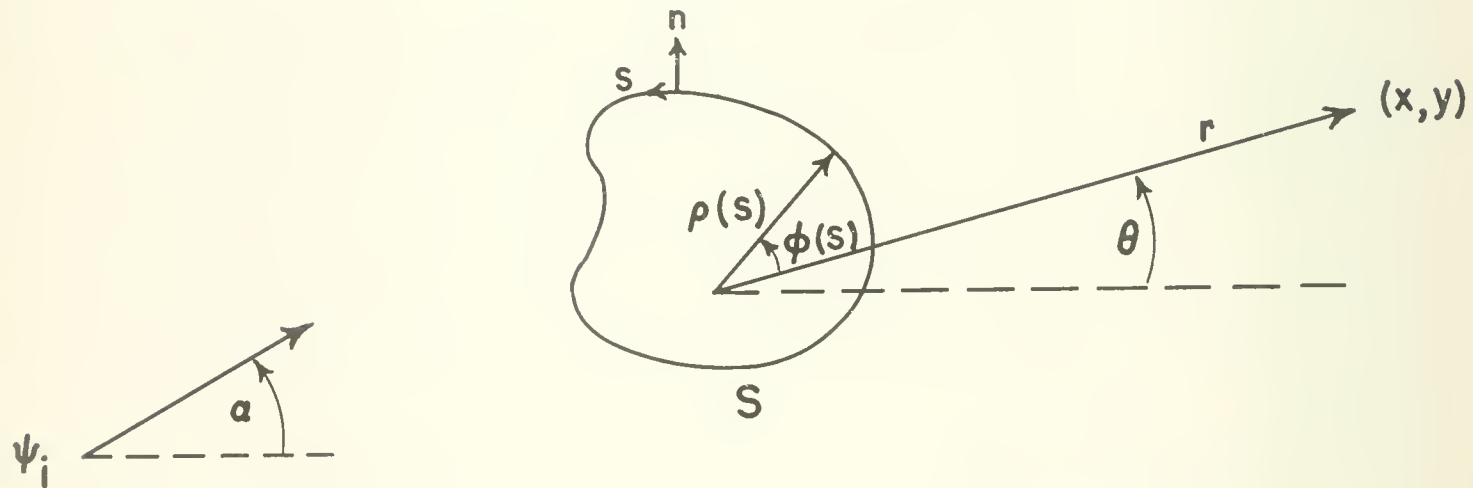


Fig. 1: Scattering of a plane wave by a cylinder. r, θ and $\rho(s), \phi(s)$ are the polar coordinates of an observation point and a point on the surface S respectively. n is the normal out of S and s is the coordinate along S .

We have in general²

2. See Pappas, ref. 1, for detailed treatment of a special case.

$$(3) \quad \Psi_c = - \int_S \left[G(r, \rho) \frac{\partial \Psi(\rho)}{\partial n} - \Psi(\rho) \frac{\partial G(r, \rho)}{\partial n} \right] ds,$$

where G is the two-dimensional free space Green's function

$$(4) \quad G(r, \rho) = \frac{i}{4} H_0^{(1)}(k|\underline{r} - \underline{\rho}|).$$

For $kr \gg 1$, $r \gg \rho$ we have

$$(4') \quad G(r, \rho) \sim \sqrt{\frac{2}{i\pi kr}} e^{ikr} \frac{i}{4} e^{-ik\rho \cos(\varphi - \theta)} \frac{1}{2} H(r) \frac{i}{4} e^{-ik\rho \cos(\varphi - \theta)}$$

and consequently (3) reduces to

$$(3') \quad \Psi_c \sim H(r) \left\{ \frac{1}{i4} \int_S e^{-ik\rho \cos(\varphi - \theta)} \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial e^{-ik\rho \cos(\varphi - \theta)}}{\partial n} \right\} ds = H(r) f(\theta, \alpha),$$

where the "scattering amplitude" $f(\theta, \alpha)$ indicates the far-field response in a direction θ to a wave incident at an angle α .

We do not intend to specify the boundary conditions and seek a solution. Rather we demonstrate the relationship of the forward scattered amplitude $f(\alpha, \alpha)$ to the absorption and scattering "cross-sections" of the cylinder.

The absorption cross-section, which is defined as the time-averaged power flow into a unit length of cylinder divided by the time-averaged flux per unit area of incident wave, may be written as

$$(5) \quad q_a = -\text{Re} \frac{1}{ik} \int_0^{2\pi} \Psi^*(r, \theta) \frac{\partial \Psi(r, \theta)}{\partial r} r d\theta = -\text{Re} \frac{1}{ik} \int_S \Psi^*(s) \frac{\partial \Psi(s)}{\partial n} ds.$$

The first integral is taken over a circle of radius r enclosing S ; the second is taken over S . From elementary considerations we know that the integrals are equal.

Similarly the scattering cross-section, which is defined as the time-averaged scattered power per unit length of cylinder divided by the time-averaged incident energy density, may be written as

$$(6) \quad q = \text{Re} \frac{1}{ik} \int_0^{2\pi} \Psi_c^*(r, \theta) \frac{\partial \Psi_c(r, \theta)}{\partial r} r d\theta = \text{Re} \frac{1}{ik} \int_S \Psi_c^*(s) \frac{\partial \Psi_c(s)}{\partial n} ds.$$

In general we have

$$(7) \quad q_a = -\operatorname{Re} \frac{1}{ik} \int (\Psi_i + \Psi_c)^* \frac{\partial(\Psi_i + \Psi_c)}{\partial n} ds = -\operatorname{Re} \frac{1}{ik} \int \left[\Psi_c^* \frac{\partial \Psi_c}{\partial n} + (\Psi_c^* \frac{\partial \Psi_c}{\partial n} + \Psi_c^* \frac{\partial \Psi_i}{\partial n} + \Psi_i^* \frac{\partial \Psi_c}{\partial n}) \right] ds$$

$$= -q - \operatorname{Re} \frac{1}{ik} \int_S \left[\Psi_i^* \frac{\partial \Psi_i}{\partial n} + \Psi_i^* \frac{\partial \Psi_c}{\partial n} \right] ds,$$

where we have added the term $\Psi_i^* \frac{\partial \Psi_i}{\partial n}$ (whose integrated value vanishes) within the square brackets. From (7) we obtain the "total cross-section"

$$(7') \quad Q \equiv q_a + q = -\operatorname{Re} \frac{1}{ik} \int_S \left[-\Psi \frac{\partial e^{-ikpcos(\varphi-\alpha)}}{\partial n} + e^{-ikpcos(\varphi-\alpha)} \frac{\partial \Psi}{\partial n} \right] ds.$$

Comparison of (7') and (3') yields the well-known relation

$$(8) \quad Q = -\frac{4}{k} \operatorname{Re} f(\alpha, \alpha);$$

thus the total cross-section of the cylinder, i.e., the normalized energy flux the cylinder obtains from the incident wave and dissipates as heat and scattered radiation, is proportional to the real part of the forward scattered amplitude³.

If there is no absorption, then Q of (8) is to be replaced by q of (6).

Letting $r \rightarrow \infty$ in the first form of q of (6) we obtain

$$(8') \quad \operatorname{Re} f(\alpha, \alpha) = -\frac{1}{2\pi} \int_0^{2\pi} |f(\Theta, \alpha)|^2 d\Theta.$$

We now use (8) to derive a corollary. If the cylinder is located at $x=0$, $y=y'$ rather than at the origin, we have

$$(9) \quad \Psi_c(y') \sim H(r') e^{iky' \sin \alpha} f(\Theta', \alpha), \quad \cos \Theta' = \frac{x}{r'} = \frac{x}{\sqrt{x^2 + (y-y')^2}};$$

the phase factor in y' is merely the phase of the incident wave at $x=0$, $y=y'$.

If $f(\Theta', \alpha)$ is a slowly varying function of y' compared to $r'-y' \sin \alpha$, then

3. Our definition of f differs by a factor of $i/4$ from that of ref. 1.

$$(10) \quad \int_{-\infty}^{\infty} \Psi_c(y') dy' \sim \frac{2}{k \cos \alpha} \Psi_1(\theta, \alpha) f(\alpha, \alpha) \quad \text{for } x > 0$$

$$\sim \frac{2}{k \cos \alpha} \Psi_1(\theta, \pi - \alpha) f(\pi - \alpha, \alpha) \quad \text{for } x < 0,$$

where the integral was evaluated by the method of stationary phase⁴. Consequently in the right half-plane we have

$$(11) \quad 2 \operatorname{Re} \Psi_1^*(\theta, \alpha) \int_{-\infty}^{\infty} \Psi_c(y') dy' = - \frac{4 \operatorname{Re} f(\alpha, \alpha)}{k \sec \alpha} = - Q \sec \alpha,$$

where the last expression follows from (8).

Eq. (11) is of physical interest in connection with the transmission of a wave through a planar uniform distribution of identical parallel cylinders. If ρ is the average number of scatterers in unit length of y -axis and if $\rho/k \cos \alpha \ll 1$, then the transmission coefficient is approximately equal to unity minus ρ times the left side of (11) (i.e., unity corresponds to the incident flux, and the left side of (11) to the interference terms of the incident and average single-scattered waves⁴). Hence by means of (11) we may write the transmission coefficient as

$$(12) \quad T \approx 1 - Q \rho \sec \alpha, \quad \rho/k \cos \alpha \ll 1.$$

This result could have been written immediately from elementary physical considerations. Q is the energy "removed" from the transmitted wave by one scatterer, and $\rho \sec \alpha$ (the number of scatterers illuminated by unit area of incident wave) times Q is the energy "missing" from unit area of the transmitted wave. (We ignore here the energy scattered into unit area of the

4. V. Twersky, On the Scattered Reflection of Scalar Waves, EM-22 (Math. Res. Grp., N.Y.U., 1950) p. 15 ff. We have shown (10) to be exact for the case of circular cylinders; see Multiple Scattering of Waves by Planar Random Distributions of Cylinders and Bosses, to be published.

transmitted wave, since it is in general small compared to Q_0 .)

3. The Reflection Case

We now consider the analogous reflection problem of a plane wave incident on a surface composed of an arbitrary cylindrical protuberance on a smooth perfectly reflecting plane as in Fig. 2.

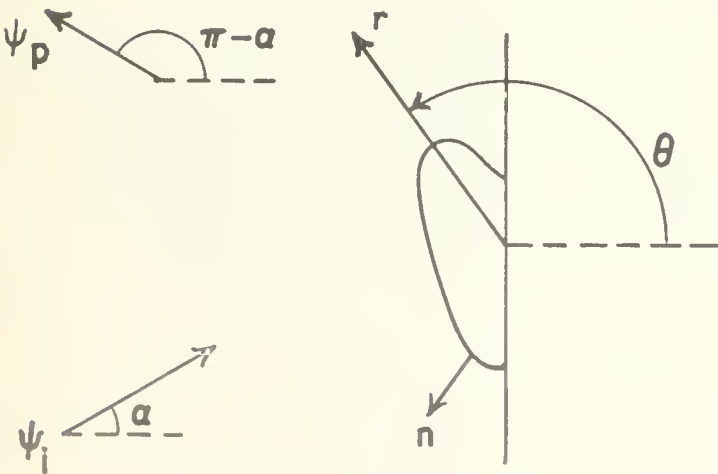


Fig. 2: Scattered reflection of a plane wave Ψ_i by a boss on a smooth perfectly reflecting plane.

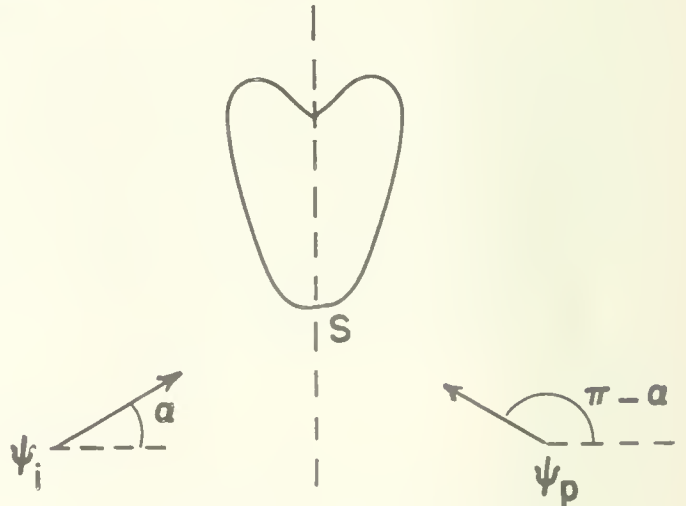


Fig. 3: The image problem: scattering of two plane waves by a cylinder.

We write the total field as

$$(13) \quad \Psi(\theta, \alpha) = \Psi_i + \Psi_p + \Psi_b,$$

where Ψ_i , the incident wave, is given by (2) and where

$$(14) \quad Y_p = \pm Y_1(\theta, \pi - \alpha) = \pm e^{ikr \cos(\theta - \pi + \alpha)}$$

is the wave reflected from the plane in the absence of the boss; the minus sign is to be taken for $Y = 0$ at $\theta = \pi/2$ (case A) and the plus sign for $\frac{\partial Y}{\partial n} = 0$ at $\theta = \pi/2$ (case B). The function Y_b , the wave scattered by the boss, satisfies the radiation condition at infinity.

The problem shown in Fig. 2 is equivalent to that shown in Fig. 3 in the left-hand plane. The problem of Fig. 3 may be formulated essentially as in Section 2; we have

$$(15) \quad Y = Y_i + Y_p + Y_b = Y_i + Y_p - \int_S \left[G \frac{\partial Y}{\partial n} - Y \frac{\partial G}{\partial n} \right] ds.$$

The value of (15) in the left-hand plane is identically (13).

The function Y_b is formally identical with Y_c of (3) and (3'), but the present two scattering amplitudes, say $g_{\pm}$, of course differ from f of (3') as well as from each other. The essential difference is one of symmetry. Thus for case A, $Y = 0$ at $\theta = \pi/2$, we must have

$$(16) \quad g_-(\theta, \alpha) = -g_-(\pi - \theta, \alpha),$$

and similarly for case B, $\frac{\partial Y}{\partial n} = 0$ at $\theta = \pi/2$, we require

$$(17) \quad g_+(\theta, \alpha) = g_+(\pi - \theta, \alpha).$$

Following the procedure of Section 2 we now seek a relation between the total cross-section and the scattering amplitude in the "specular direction of reflection" (see Fig. 2) $g(\pi - \alpha, \alpha)$.

The absorption and scattering cross-sections are given by (5) and (6), except that the right-hand sides of the equations are to be divided by 2; this normalization takes into account the fact that we deal here with two incident plane waves, and facilitates extension of the final results to the problem of Fig. 2. We now have in analogy to (7)

$$\begin{aligned}
 (18) \quad q_a &= -\operatorname{Re} \frac{1}{2ik} \int_S (\Psi_i + \Psi_p + \Psi_b)^* \frac{\partial}{\partial n} (\Psi_i + \Psi_p + \Psi_b) ds \\
 &= -\operatorname{Re} \frac{1}{2ik} \int_S \left\{ \Psi_b^* \frac{\partial \Psi_b}{\partial n} + \left[\Psi_b^* \frac{\partial (\Psi_i + \Psi_p)}{\partial n} + (\Psi_i + \Psi_p) \frac{\partial \Psi_b}{\partial n} \right] \right\} ds \\
 &= -q - \operatorname{Re} \frac{1}{2ik} \int_S \left[\Psi^* \frac{\partial (\Psi_i + \Psi_p)}{\partial n} + (\Psi_i + \Psi_p)^* \frac{\partial \Psi}{\partial n} \right] ds,
 \end{aligned}$$

where we have added the term $(\Psi_i + \Psi_p)^* \frac{\partial (\Psi_i + \Psi_p)}{\partial n}$ (whose integrated value vanishes) within the square brackets. Hence

$$(19) \quad Q = q_a + q = -\operatorname{Re} \frac{1}{2ik} \int_S \left[-\Psi \frac{\partial (\Psi_i + \Psi_p)^*}{\partial n} + (\Psi_i + \Psi_p)^* \frac{\partial \Psi}{\partial n} \right] ds.$$

For case A we have

$$(20) \quad \Psi_i^*(s) + \Psi_p^*(s) = e^{-ikpcos(\theta-\alpha)} - e^{-ikpcos(\theta-\pi+\alpha)}.$$

Hence from the definition (3') and the symmetry property (16) we obtain

$$(21) \quad Q_- = -\frac{2}{k} \operatorname{Re} [g_-(\alpha, \alpha) - g_-(\pi-\alpha, \alpha)] = \frac{4}{k} \operatorname{Re} g_-(\pi-\alpha, \alpha).$$

For case B we change the sign of the second term of (20) and obtain

$$(22) \quad Q_+ = -\frac{2}{k} \operatorname{Re} [g_+(\alpha, \alpha) + g_+(\pi-\alpha, \alpha)] = -\frac{4}{k} \operatorname{Re} g_+(\pi-\alpha, \alpha).$$

(21) and (22) are formally identical except for the sign. Hence, for uniformity, we now define $g_-(\theta, \alpha)$ to be the negative of the function defined in (3') (or else we may simply change the signs of Ψ_i and Ψ_p for case A). On suppressing the subscript $\pm$ we may write for either case

$$(23) \quad Q = q_a + q = -\frac{4}{k} \operatorname{Re} g(\pi-\alpha, \alpha),$$

which is the desired relation.

Note that we have not simply added the result for a symmetrical cylinder with Ψ_i incident to that for Ψ_p incident. In that case we would have used (8) and obtained $-\frac{4}{k} \operatorname{Re} [f(\alpha, \alpha) + f(\pi-\alpha, \pi-\alpha)] = Q_i + Q_p = -\frac{8}{k} \operatorname{Re} f(\alpha, \alpha) = 2Q_1$; however, this has no connection with the present problem.

In terms of the result for a single plane wave we may rewrite (22) as

$$Q_+ = -\frac{4}{K} \operatorname{Re}[f(\pi-\alpha, \alpha) + f(\pi-\alpha, \pi-\alpha)],$$

where the first function is evaluated at the specular angle of reflection with respect to the plane of symmetry, and the second is evaluated in the forward scattered direction. The difference between the present problem and that of Section 2 is clear for the case of non-absorption. In analogy to (8') we now obtain

$$(24) \quad \operatorname{Re} g(\pi-\alpha, \alpha) = -\frac{1}{4\pi} \int_0^{2\pi} [g(\theta, \alpha)]^2 d\theta = -\frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} [g(\theta, \alpha)]^2 d\theta.$$

Hence in terms of the amplitudes corresponding to a single plane wave, we have for case B

$$(24') \quad \operatorname{Re}[f(\pi-\alpha, \alpha) + f(\pi-\alpha, \pi-\alpha)] = -\frac{1}{4\pi} \int_0^{2\pi} [f(\theta, \alpha) + f(\theta, \pi-\alpha)]^2 d\theta.$$

For case A and $-\Psi_1$ incident we simply change the signs of the $f(\theta, \alpha)$ terms. Clearly we have not added energies which are quadratic. Equation (8') enables us to correlate the second term on the left with only the $|f(\theta, \alpha)|^2$ term on the right.

Eq. (23) was derived for the problem of Fig. 3. However, it also holds for the problem of Fig. 2 by virtue of our normalization; i.e., in Fig. 2 we have only one incident wave (this eliminates the factor of 1/2 we introduced in q_a of (18)), but we integrate only over $\pi/2 < \theta < 3\pi/2$ (this restores the factor of 1/2). Hence in using (23), we take q_a and q as in (5) and (6) and either introduce the factor 1/2 or else integrate only over the left-hand plane.

Eq. (23) is a reflection theorem analogous to the "transmission theorem" of (8). The present theorem states that the total energy cross-section of an arbitrary boss on a perfectly reflecting plane is proportional to the real part of the scattering amplitude evaluated

in the direction of specular reflection of the plane.

We now use (23) to derive a corollary analogous to (11). We proceed as for (9) and (10) with the Ψ_c replaced by Ψ_b and f replaced by g ; here only the result of (10) for $x < 0$ applies. Hence we find

$$(25) \quad 2 \operatorname{Re} \Psi_p^* \int_{-\infty}^{\infty} \Psi_b(y') dy' \sim - \frac{4 \operatorname{Re} g(\pi-\alpha, \alpha)}{k \cos \alpha} = - Q \sec \alpha$$

where $\Psi_p = \Psi_i(\theta, \pi-\alpha)$, in accordance with the convention introduced for (23). This theorem is of physical interest in connection with reflection from a uniform distribution of bosses on a plane (i.e., a "striated surface"). In analogy to the transmission coefficient of (12) we now have the reflection coefficient

$$(26) \quad R \approx 1 - Q \rho \sec \alpha, \quad \rho/k \cos \alpha \ll 1,$$

where $Q \rho \sec \alpha$ is the energy "missing" from unit area of the wave reflected at the specular angle of the plane.

The present results can be readily extended to the analogous three-dimensional problems.

Reflection Coefficients for Certain Rough Surfaces*

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 (Received February 23, 1953)

IN a recent paper¹ we considered the scattered reflection of electromagnetic waves from perfectly conducting surfaces composed of either semicylindrical or hemispherical bosses on a smooth plane. In this letter we obtain reflection coefficients for these surfaces by equating the scattered energy resulting from unit area of incident wave with the energy "missing" from unit area of the wave reflected at the specular angle. We then indicate how the identical results may be obtained analytically. We assume single scattering but will indicate the range of validity of the results on the basis of our multiple scattering treatment of the cylindrical case.²

The incident wave is specified by $\exp[-ikr \cos(\theta - \alpha) - i\omega t]$, where α and θ , the angles of incidence and observation, are measured from the normal of the plane. The incident polarization amplitudes are taken to be unity perpendicular and parallel to the plane of incidence; the corresponding scattered fields are indicated by the subscripts $\perp$ and $\parallel$. We first consider the two-dimensional cylindrical problem and then the spherical problem. The acoustic case is also treated.

The far-field form of the wave scattered by a semicylindrical boss of radius $a \ll k^{-1} = \lambda/2\pi$ is specified by¹

$$\begin{Bmatrix} E_{\perp} \\ E_{\parallel} \end{Bmatrix} = \left(\frac{\pi i}{2kr}\right)^{1/2} (ka)^2 e^{ikr} \begin{Bmatrix} 2 \cos \alpha \cos \theta \\ -1 - 2 \sin \alpha \sin \theta \end{Bmatrix}. \quad (1)$$

The number of scatterers excited by unit area of incident wave is $\rho \sec \alpha$, where ρ is the number of bosses in unit length of surface. Hence the normalized energy flux single scattered from unit area of incident wave, or the "scattering cross section of the surface per unit area of incident wave," is (for either component)

$$\sigma = q \rho \sec \alpha = \rho \sec \alpha \int_{-\pi/2}^{\pi/2} |E|^2 r d\theta, \quad r \rightarrow \infty, \quad (2)$$

where q is the scattering cross section per unit length of a single boss. For $\rho/k \ll 1$ we take σ to be the energy "missing" from unit area of the wave reflected at the specular angle $\theta = -\alpha$. Thus we write the reflection coefficients as

$$R = 1 - \sigma \approx e^{-\sigma}, \quad (3)$$

where it is understood that $\sigma \ll 1$; specifically we have

$$\begin{Bmatrix} R_{\perp} \\ R_{\parallel} \end{Bmatrix} = 1 - \frac{\rho}{k} \pi^2 (ka)^4 \begin{Bmatrix} \cos \alpha \\ (1 + 2 \sin^2 \alpha)/2 \cos \alpha \end{Bmatrix}. \quad (4)$$

$R_{\parallel}$ of Eq. (4) breaks down near grazing incidence ($\alpha \rightarrow \pi/2$) as a consequence of multiple scattering.³

For the analogous problem of the hemispherical boss we have¹

$$\begin{Bmatrix} E_{\perp} \\ E_{\parallel} \end{Bmatrix} = \frac{k^2 a^2}{r} e^{ikr} \begin{Bmatrix} \cos \alpha (\cos \theta \cos \phi \cos \varphi + \theta \sin \varphi) \\ \phi \cos \theta \sin \varphi - \theta (2 \sin \theta \sin \alpha + \cos \varphi) \end{Bmatrix}, \quad (5)$$

where θ and ϕ are unit vectors; θ is the polar angle and φ is measured from the plane of incidence. The scattering cross section of the surface per unit area of incident wave is

$$\sigma = q \rho \sec \alpha = \rho \sec \alpha \int_0^{2\pi} d\varphi \int_0^{\pi/2} |E|^2 r^2 \sin \theta d\theta, \quad (2')$$

where q is the scattering cross section of a single boss and ρ is the number of scatterers in unit area of surface. The reflection coefficients are thus

$$\begin{Bmatrix} R_{\perp} \\ R_{\parallel} \end{Bmatrix} = 1 - \frac{\rho 4\pi (ka)^6}{k^2 3} \begin{Bmatrix} \cos \alpha \\ (1 + 4 \sin^2 \alpha)/\cos \alpha \end{Bmatrix}, \quad (6)$$

where $\rho/k^2 \ll 1$. $R_{\parallel}$ breaks down as $\alpha \rightarrow \pi/2$.

We now consider the corresponding acoustic problems. For the

cylindrical case $E_{\perp}$ and $E_{\parallel}$ of Eq. (1) are the velocity potentials for a free and rigid surface respectively; consequently, the results of Eq. (4) apply. For the spherical case this correspondence does not hold; instead the scattered velocity potentials for the single boss are given by¹

$$\begin{Bmatrix} \psi_f \\ \psi_r \end{Bmatrix} = \frac{k^2 a^3}{r} e^{ikr} \begin{Bmatrix} 2 \cos \alpha \cos \theta \\ (-2 - 3 \sin \alpha \sin \theta \cos \varphi)/3 \end{Bmatrix}. \quad (7)$$

Consequently,

$$\begin{Bmatrix} R_f \\ R_r \end{Bmatrix} = 1 - \frac{\rho 8\pi (ka)^6}{k^2 3} \begin{Bmatrix} \cos \alpha \\ [1 + (3/4) \sin^2 \alpha]/3 \cos \alpha \end{Bmatrix}, \quad (8)$$

where $\rho/k^2 \ll 1$; R_r breaks down as $\alpha \rightarrow \pi/2$.

The intuitive procedure we used above can be justified and generalized analytically for a large class of nonabsorbing surfaces. Consider the two-dimensional problem of a wave incident on an arbitrary configuration of identical (not necessarily circular) bosses on an infinite plane. We calculate the total time-average Poynting flux for a particular surface (i.e., a particular configuration) and average the flux over the ensemble of surfaces. If the distribution is uniform, then the normalized average reflected flux to lowest order in ρ (actually $\rho/k \cos \alpha$) is

$$\langle I \rangle = \left\{ 1 + 2\rho R_e \left[E_p^* \int E(y') dy' \right] \right\} n + \rho \int |E(y')|^2 r_0' dy', \quad (9)$$

where ρ is the average number of elements in unit length, $E(y')$ is the single-scattered wave of a boss located at $x=0$, $y=y'$, and E_p is the wave reflected from the plane in the absence of the bosses. $\mathbf{n} = \mathbf{i} \cos \alpha - \mathbf{j} \sin \alpha$ is a unit vector in the specular direction, and $\mathbf{r}_0' = \mathbf{i} \cos \theta' + \mathbf{j} \sin \theta'$ is a unit vector from $x=0$, $y=y'$ to the observation point x, y . We now show that the interference term of Eq. (9) equals σ of Eq. (2) for a boss of arbitrary cross section.

Since the component of the reflected flux normal to the surface must equal $\cos \alpha$ for a nonabsorbing surface, we obtain from Eq. (9) the relation

$$-2 \cos \alpha R_e \left[E_p^* \int E(y') dy' \right] = \int |E(y')|^2 \cos \theta' dy', \quad (10)$$

where $\cos \theta' = x/r' = x[x^2 + (y-y')^2]^{-1/2}$. We now make use of $d\theta' = -\cos \theta' dy'/r'$ and change to the variable θ' on the right side of Eq. (10); then, if the limits in the integrals of Eq. (10) are infinite, we obtain

$$-2 \cos \alpha R_e \left[E_p^* \int_{-\infty}^{\infty} E(y') dy' \right] = \int_{-\pi/2}^{\pi/2} |E(\theta')|^2 r_0' d\theta' = q. \quad (11)$$

Thus the terms arising from interference of the specular and scattered waves in Eq. (9) are equal to $-\rho \sec \alpha$ times the scatter-cross section of a single boss, i.e., to σ . [We have also proved theorem Eq. (11) directly for a single boss without recourse to probability considerations.]

For the special case of a circular boss we employ the complete form of $E(y')$ and find that either the left or right side of Eq. (11) equals

$$\begin{Bmatrix} q_{\perp} \\ q_{\parallel} \end{Bmatrix} = \frac{8}{k} \sum_{n=-\infty}^{\infty} \begin{Bmatrix} |A_n|^2 \sin^2 n(\alpha + \pi/2) \\ |B_n|^2 \cos^2 n(\alpha + \pi/2) \end{Bmatrix}, \quad |A_n|^2 = \frac{J_n^2}{J_n^2 + N_n^2},$$

$$|B_n|^2 = \frac{J_n'^2}{J_n'^2 + N_n'^2}, \quad (12)$$

where the argument of the cylindrical functions is ka and where the prime indicates differentiation with respect to ka . The above may be readily extended to the three-dimensional case.

We now show that Eq. (3) is a consequence of Eq. (9) and Eq. (11). We may apply $\langle I \rangle$ of Eq. (9) to a practical set-up, provided that $\rho/k \cos \alpha \ll 1$ and provided that the width and height of the bosses are small compared to ρ^{-1} ; the range of integration is to be taken as the illuminated region. If the incident beam width, say d , is very much larger than all wavelengths involved, then we may, in general, use infinite limits in the $\mathbf{n}$ component: because of the exponent of $E(y')$ the main contribution of the integral arises from the immediate vicinity of the specular or stationary point and is consequently practically independent of the width of the illuminated region.⁵ Hence for the $\mathbf{n}$ component we may use Eq. (11) and the definition $\sigma = q\rho \sec \alpha$ to write the leading term of Eq. (9) as $(1-\sigma)\mathbf{n}$. For the $\mathbf{r}_0'$ component (the "incoherent scattering"), however, the integrand is slowly varying, and the integration should be taken only over the illuminated region. If this region is small compared to the distance r of the observation point from its center (i.e., if $d \sec \alpha / r \ll 1$), then we may write this term as $\rho d \sec \alpha |\mathbf{E}(r, \theta)|^2 \mathbf{r}_0$. The function $|\mathbf{E}(r, \theta)|^2$ is the normalized scattered energy flux of a boss at the center of the illuminated region, $\mathbf{r}_0$ is the unit vector from this boss to the observation point, and $\rho d \sec \alpha$ is the number of illuminated bosses. (For small circular bosses $|\mathbf{E}|^2$ may be obtained directly from Eqs. (1), (5), and (7).)

Hence we may write

$$\langle I \rangle \approx (1-\sigma)\mathbf{n} + \rho d \sec \alpha |\mathbf{E}|^2 \mathbf{r}_0. \quad (13)$$

This expression is to be used as follows: at the specular angle we ignore the term in $|\mathbf{E}|^2$ (i.e., the energy scattered *into* the reflected beam), since it is only of the order of d/r times σ ; thus we have

$$\langle I \rangle \cdot \mathbf{n} \approx 1 - \sigma = R, \quad \theta = -\alpha. \quad (14)$$

We have therefore justified the use of Eq. (3), provided that $\rho/k \ll 1$ and $\alpha \rightarrow \pi/2$. On the other hand, if θ is not near $-\alpha$, we ignore the first term of Eq. (13) and use

$$\langle I \rangle \cdot \mathbf{r}_0 \approx \rho d \sec \alpha |\mathbf{E}|^2, \quad (15)$$

i.e., outside of the reflected beam we consider only the incoherent scattering.

We note that the results can be extended to the case of absorbing bosses by replacing σ by $\sigma + \sigma_a = (q + q_a)\rho \sec \alpha$, where

q_a is the absorption cross section. However, it is simpler for this case to obtain R directly from the $\mathbf{n}$ component of Eq. (9). Thus if we write $E(y') = (2/i\pi k r) \frac{1}{2} e^{ik(r'-y' \sin \alpha)} f(\theta', \alpha)$, where $f(\theta', \alpha)$ is a slowly varying function of y' compared to the exponent, then² $\int E(y') dy' \approx E_p(2/k \cos \alpha) \times f(-\alpha, \alpha)$. Consequently, we have

$$\langle I \rangle \cdot \mathbf{n} \approx 1 + (4\rho/k \cos \alpha) \operatorname{Re}[f(-\alpha, \alpha)] = R, \quad (16)$$

where $f(-\alpha, \alpha)$ is the value of the scattering amplitude at the specular angle. (Most generally we express $f(\theta, \alpha)$ as an integral over the surface of the boss.) It can be shown that R of Eq. (16) is identically $1 - (\sigma + \sigma_a)$. The leading term of R of Eq. (16) involving ka (or an equivalent parameter) is of the lowest order of magnitude found in E , e.g., $R_{11} = 1 - \mathcal{O}(ka)^2$ for the circular cylindrical case, etc. [Compare with Eq. (14).]

We have assumed in the above that the incident beam is perfectly collimated and also that the pick-up is highly directional. In practice, one would take into account the radiation characteristics of both the source and pick-up. Note also that since our expressions are averages they should be compared with the average of measurements over a large number of appropriate surfaces.

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¹ V. Twersky, J. Appl. Phys. 22, 825 (1951); see also J. Acoust. Soc. Am. 22, 539 (1950); 23, 336 (1951) for the acoustic case.

² V. Twersky, *Multiple Scattering of Waves by Planar Random Distributions of Parallel Cylinders and Bosses*. Mathematics Research Group, New York University (to be published).

³ We found (see reference 2) that both R_{11} and $R_{11} - 1$ as $\alpha \rightarrow \pi/2$; furthermore, the total intensity $\rightarrow 0$ as $-\theta = \alpha \rightarrow \pi/2$ regardless of polarization. We used the probability distribution functions of liquid state theory to define the ensemble of surfaces, the "randomness" of the distribution being specified by the ratio of the average separation of two scatterers to their minimum separation. We found the behavior at $-\theta = \alpha \rightarrow \pi/2$ to be independent of this ratio (and also of the dielectric properties of the bosses and of the value of ka), provided that the minimum separation was large compared to λ and a .

⁴ Note that for the circular boss, the $\mathbf{n}$ component of Eq. (9) yields the results of Eq. (4), provided that terms to the order of $(ka)^4$ are retained in $E(y')$ and provided that the limits on the integral are infinite. We did not get the present result in reference 1 since we kept only the $(ka)^2$ terms of $E(y')$, i.e., the results of reference 1 are, in general, correct only to the order of $(ka)^2$ or $(ka)^4$.

⁵ If this condition is not fulfilled, then we may use the radial energy fluxes of reference 1 for the finite distributions to the order of $(ka)^2$ or $(ka)^4$. For this case the flux may show a minimum at the specular angle, i.e., the width of the "interference beam" is narrower than the width of the specular beam represented by 1 in Eq. (9), and consequently, the value at $\theta = -\alpha$ may be smaller than the values in the immediate vicinity of $-\alpha$.

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